explosion, and that of the heat source may be obtained as self-similar solutions of the second kind if defining parameters of the pre-self-similar statement of the problem are "unluckily" selected. The possibility of obtaining these solutions in the form of self-similar solutions of the first kind is related to the selection of energy E and total heat Q as the determining parameters which, by virtue of the corresponding integral conservation laws, do not change in time).

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ON THE PERTURBATION OF A FILTRATION FLOW BY A SINGLE CRACK

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1. We consider the problem of steady filtration in a thin layer in the presence of a single crack.

The flow in a crack which can be regarded as a piecewise-smooth line Γ in studying the external filtration field is describable by means of the equations of a lubricant layer, i.e. the pressure can be assumed constant within each cross section but different at each one of them; $p^-(s) = p^+(s) = p(s)$; the fluid velocity u_0 inside the crack can be assumed to have a parabolic profile,

$$u_0 = \frac{n^2 - 2k - h^2}{2\mu}\frac{\partial p}{\partial s} \tag{1.1}$$

Here n is the normal to the crack axis, $2h\ (s)$ is the width of the crack at the cross section $M\ (s)$, μ is the viscosity of the fluid, and k is the permeability of the porous medium.

The volume rate of the fluid flow through the cross section $M\ (s)$ is given by

$$Q\ (s) = \int_{-h}^{h} u_0 dn = \frac{2h\,(h^2 + 3k)}{3\mu}\frac{\partial p}{\partial s} \tag{1.2}$$

From the condition of conservation of the mass of filtered fluid in each crack element MM' we obtain the following condition along the crack:

$$\frac{\partial p^+}{\partial n} - \frac{\partial p^-}{\partial n} = \frac{\mu}{k}\frac{\partial Q}{\partial s} \tag{1.3}$$

Making use of Eq. (1.2), we can rewrite this condition as

$$\frac{\partial p^+}{\partial n} - \frac{\partial p^-}{\partial n} + \frac{\partial}{\partial s}\left(\delta \frac{\partial p}{\partial s}\right) = 0, \qquad \delta(s) = \frac{2h(h^2 + 3k)}{3k} \geqslant 2h(s) \geqslant 0 \tag{1.4}$$

Let us introduce the complex potential

$$W(z) = \varphi + i\psi, \qquad \varphi = -kp / \mu$$

Here ψ is the stream function. Integrating over s, we obtain

$$\delta \frac{\partial \varphi}{\partial s} - c_0 = \psi^+ - \psi^-, \qquad c_0 = \delta(0)\varphi'(0) \tag{1.5}$$

Condition (1.4) and the analysis to follow are also valid for isothermal gas filtration provided we replace the pressure p by its square.

The author of [1] obtained a condition of the (1.5) type on the basis of other considerations.

2. Let us consider the case where the crack axis coincides with the segment Γ,

$$\zeta = z_0 s, \qquad z_0 = ae^{i\beta}, \qquad |s| < 1$$

Following [1], we shall attempt to find the complex potential in the form

$$W(z) = F(z) + \frac{1}{2\pi}\int_\Gamma \frac{\omega(t)\,dt}{t - z} \tag{2.1}$$

Here $F(z)$ is some analytic function whose point singularities do not lie on the crack; $\omega(t)$ is the required function which satisfies the Hölder condition at the segment Γ.

Applying the Sochocki formulas [2] to the function $W(z)$ as $z \to \zeta \in \Gamma$ and making use of boundary condition (1.5) for $c_0 = 0$ [1], we obtain a functional equation for determining the discharge function,

$$\omega(s) = \psi^+ - \psi^-, \ \omega(-1) = \omega(1) = 0$$

$$a\frac{\omega(s)}{\delta(s)} = \operatorname{Re} F'(\zeta) z_0 + \frac{1}{2\pi}\frac{d}{ds}\int_{-1}^{1} \frac{\omega(\sigma)\,d\sigma}{\sigma - s}, \qquad |s| < 1 \tag{2.2}$$

The inverse problem (that of finding the function $\delta(s)$ on the basis of a given function $\omega(s)$ and using it to construct the corresponding flow) is always solvable. The case $\omega(s) = \omega_0(1 - s^2)$ is investigated in [1].

We shall solve the direct problem assuming the existence of the derivative $\omega'(s)$ which satisfies the Hölder condition in the interval $(-1, 1)$. Under this assumption and with allowance for the conditions $\omega(-1) = \omega(1) = 0$, we can rewrite Eq. (2.2) as

$$2\pi a \frac{\omega(s)}{\delta(s)} - \int_{-1}^{1} \frac{\omega'(\sigma)\,d\sigma}{\sigma - s} = 2\pi \operatorname{Re} F'(\zeta) z_0, \qquad |s| < 1 \tag{2.3}$$

The function $\omega(s)$ is therefore that solution of the integrodifferential equation (2.3) which satisfies the homogeneous boundary conditions. This equation is the familiar Prandtl equation of the theory of wings of finite span investigated in [3–5].

In the special case of the function $\delta(s) = \sqrt{1 - s^2} / \rho(s)$, where $\rho(s)$ is an even analytic positive-valued function on the segment $[-1, 1]$, Eq. (2.3) is equivalent to the integral equation [4]

$$\omega(s)=\omega(0)\cos\theta(s)+\int_{-1}^{1}K_0(s,\sigma)\,\omega(\sigma)\,d\sigma+g_0(s) \qquad (2.4)$$

$$g_0(s)=-2\int_0^s\sin[\theta(t)-\theta(s)]\,\mathrm{Re}\,F'(\zeta)\,z_0dt+$$

$$+\frac{2}{\pi}\int_0^s\frac{\cos[\theta(t)-\theta(s)]}{\sqrt{1-t^2}}\,dt\int_{-1}^{1}\frac{\sqrt{1-\sigma^2}}{\sigma-t}\,\mathrm{Re}\,F'(\zeta)\,z_0d\sigma$$

$$K_0(s,\sigma)=-\frac{2a}{\pi}\int_0^s\frac{\cos[\theta(t)-\theta(s)]}{\sqrt{1-t^2}}\,\frac{\rho(\sigma)-\rho(t)}{\sigma-t}\,dt,\qquad \theta(s)=2a\int_0^s\frac{dt}{\delta(t)},\quad |s|\leqslant 1$$

If $\cos\theta(1)\neq 0$, then we make use of the conditions $\omega(-1)=\omega(1)=0$ to obtain the Fredholm integral equation

$$\omega(s)-\int_{-1}^{1}K(s,\sigma)\,\omega(\sigma)\,d\sigma=g(s) \qquad (2.5)$$

$$g(s)=g_0(s)-\frac{g_0(1)\cos\theta(s)}{\cos\theta(1)},\qquad K(s,\sigma)=K_0(s,\sigma)-\frac{K_0(1,\sigma)\cos\theta(s)}{\cos\theta(1)} \qquad (2.6)$$

For $\cos\theta(1)=0$ we obtain the system

$$\omega(s)=\omega(0)\cos\theta(s)+\int_{-1}^{1}K_0(s,\sigma)\,\omega(\sigma)+g_0(s) \qquad (2.7)$$

$$\int_{-1}^{1}K_0(1,\sigma)\,\omega(\sigma)\,d\sigma+g_0(1)=0$$

The undetermined constant appearing in the first equation of (2.7) must generally be determined from the second equations of (2.7). If this constant is not determined from the latter equation, then it must be chosen in such a way that the solution of the first equation of system (2.7) is the solution of Eq. (2.3).

The kernels of Eqs. (2.6), (2.7) become degenerate if $\rho(s)=\sqrt{1-s^2}/\delta(s)$ is a rational function. The required function $\omega(s)$ can then be expressed explicitly in quadratures [4].

3. Let $\delta(s)=b_0\sqrt{1-s^2}$, $b_0=\mathrm{const}$; then $\rho(s)=\mathrm{const}$; $K_0(s,\sigma)=0$. From (2.4) we obtain

$$\omega(s)=\omega(0)\cos\theta(s)+g_0(s),\quad \theta(s)=\chi\arcsin s,\quad \chi=2a/b_0 \qquad (3.1)$$

Formula (3.1) enables us to construct the solution of Eq. (2.3) in explicit form for any function $F'(z)$.

For example, if $F'(z)=V$, then $\mathrm{Re}F'(\zeta)\,z_0=Va\cos\beta$, and some simple operations bring us to the solution

$$\omega(s)=\frac{2Va\cos\beta}{1+\chi}\sqrt{1-s^2} \qquad (3.2)$$

which is also valid in the case $\cos\theta(1)=\cos(a\pi/b_0)=0$.

Formulas (2.1) and (3.2) yield the following expression for the complex potential $W(z)$:

$$W(z)=Vz+\frac{Ve^{-i\beta}\cos\beta}{1+\chi}\left[\sqrt{z^2-z_0^2}-z\right]+c,\qquad c=\mathrm{const} \qquad (3.3)$$

In the case $\beta=0$ we can take the limit as $z\to x\pm i0$ to obtain the following expression for the complex velocity:

$$\frac{dW^{\pm}}{dz} = u^{\pm} - iv^{\pm} = \begin{cases} V - V(1+\chi)^{-1}[1 \pm ix(a^2 - x^2)^{-1/2}] & (|x| < a) \\ V - V(1+\chi)^{-1}[1 - |x|(x^2 - a^2)^{-1/2}] & (|x| > a) \end{cases} \tag{3.4}$$

From the expression for $v^{\pm}$ in (3.4) we see that the crack absorbs the layer fluid for $-a \leqslant x \leqslant 0$ and supplies fluid to the layer for $0 \leqslant x \leqslant a$.

If $F(z) = A \ln(z - b)$, $b \overline{\in} \Gamma$, then the function $g_0(s)$ in formula (3.1) for $\omega(s)$ becomes

$$g_0(s) = -2 \int_0^s \sin[\theta(t) - \theta(s)] \operatorname{Re} \frac{Az_0}{z_0 t - b} dt +$$

$$+ \frac{2}{\pi} \int_0^s \frac{\cos[\theta(t) - \theta(s)]}{\sqrt{1-t^2}} dt \int_{-1}^{1} \frac{\sqrt{1-\sigma^2}}{\sigma - t} \operatorname{Re} \frac{Az_0}{z_0\sigma - b} d\sigma \tag{3.5}$$

For $\cos\theta(s) \neq 0$ we have

$$\omega(s) = g_0(s) - \frac{g_0(1)\cos\theta(s)}{\cos\theta(1)} \tag{3.6}$$

Contour integration yields the following expression for the discharge q of a borehole at the point $z = b$:

$$q = \frac{2\pi k}{\mu} \frac{p_* - p_0}{\ln(r_*/r_0) - \lambda}$$

$$\lambda = \frac{1}{2\pi} \operatorname{Re}\left(z_0 \int_{-1}^{1} \frac{\omega_1(s)\,ds}{z_0 s - b}\right), \qquad \omega_1(s) = \frac{\omega(s)}{A} \tag{3.7}$$

Here p_0 is the pressure at the working contour of a borehole of radius r_0; P_* is the pressure at a circular feed contour of radius r_* with its center at the point $z = b$.

4. In the more general case where the function $1/\delta(s)$ is integrable on the segment $[-1, 1]$ and where $\delta(s) > 0$ for $|s| < 1$, we can express the required function $\omega(s)$ in the form

$$\omega(s) - \int_{-1}^{s} \omega'(\sigma)\,d\sigma = \int_{-1}^{1} \chi(s,\sigma)\,\omega'(\sigma)\,d\sigma, \qquad \chi(s,\sigma) = \begin{cases} 1 & (\sigma \leqslant s) \\ 0 & (\sigma > s) \end{cases} \tag{4.1}$$

Making use of expression (4.1), we can rewrite (2.3) as

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{\omega'(\sigma)\,d\sigma}{\sigma - s} = \frac{a}{\delta(s)} \int_{-1}^{1} \chi(s,\sigma)\,\omega'(\sigma)\,d\sigma - F_1(s) \tag{4.2}$$

$$F_1(s) = \operatorname{Re} F'(\zeta)\,z_0, \qquad |s| < 1$$

Assuming that the right side of Eq. (4.2) is known, we make use of the inversion formula for an integral with a Cauchy kernel to obtain [2, 6]

$$\omega'(s) = -\frac{1}{\sqrt{1-s^2}} \frac{2}{\pi} \int_{-1}^{1} \left[\frac{a}{\delta(\tau)} \int_{-1}^{1} \chi(\tau,\sigma)\,\omega'(\sigma)\,d\sigma - F_1(\tau)\right] \frac{\sqrt{1-\tau^2}}{\tau - s} d\tau \tag{4.3}$$

The above assumptions concerning the functions $\omega'(\sigma)$ and $\delta(\tau)$ in the iterated integral enable us to alter the order of integration [6].

We have

$$\omega'(s) = \frac{1}{\sqrt{1-s^2}} \left[\int_{-1}^{1} K_1(s,\sigma)\,\omega'(\sigma)\,d\sigma + f_0(s)\right], \qquad |s| < 1 \tag{4.4}$$

$$K_1(s,\sigma) = \frac{2a}{\pi} \int_1^{\sigma} \frac{\sqrt{1-\tau^2}}{\delta(\tau)} \frac{d\tau}{\tau - s}, \qquad f_0(s) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-\tau^2}\,F_1(\tau)\,d\tau}{\tau - s} \tag{4.5}$$

From (4. 5) we see that the kernel $K_1(s, \sigma)$ has a logarithmic singularity for $s = \sigma$ [2, 6]. Introducing the notation $\omega'(s) = \omega_0'(s) / \sqrt{1 - s^2}$, we can rewrite Eq. (4. 4) as

$$\omega_0'(s) = \int_{-1}^{1} \frac{K_1(s, \sigma)}{\sqrt{1-\sigma^2}} \omega_0'(\sigma)\, d\sigma + f_0(s), \qquad |s| < 1 \tag{4.6}$$

In order to eliminate the "fixed" infinity at the ends of the segment [-1, 1], we make the substitution of variables

$$\sigma = \sin\sigma_1, \qquad s = \sin s_1 \tag{4.7}$$

Denoting $\omega'_0[s(s_1)]$ by $\omega_0'(s_1)$, we find from (4. 6), (4. 7) that

$$\omega_0'(s_1) = \int_{-\pi/2}^{\pi/2} K(s_1, \sigma_1)\, \omega_0'(\sigma_1)\, d\sigma_1 + f(s_1), \qquad |s_1| < \pi/2 \tag{4.8}$$

We can proceed in the same way in the case of a fixed integrable infinity of an order $\alpha < 1$.

By virtue of the foregoing statements, Eq. (4. 8) has a logarithmic (movable) singularity only and is therefore a quasi-Fredholm integral equation. The Fredholm alternative, the theorem on solvability conditions, and the other results of the Fredholm theory are valid for this equation (after the first iteration).

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